Numbers

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Outline

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2. Hexadecimal numbers
3. Integers
4. Floating point numbers
5. Converting decimal numbers to floats
6. Floating point mathematics

Binary numbers

- Decimal place value system

$$
\begin{aligned}
15301201 &= 1 * 10^7 + 5 * 10^6 + 3 * 10^5 + 10^3 + 2 * 10^2 + 1 \\
&= 10000000 + 5000000 + 300000 + 1000 + 200 + 1 \\
&= 15301201
\end{aligned}
$$

- Binary place value system

$$
\begin{aligned}
10101111 &= 2^7 + 2^5 + 2^3 + 2^2 + 2 + 1 \\
&= 128 + 32 + 8 + 4 + 2 + 1 \\
&= 175
\end{aligned}
$$

Bit numbering

bit value	1	0	1	0	1	1	1	1
bit position	7	6	5	4	3	2	1	0

- The least significant bit of a byte is bit 0
- The most significant bit is bit 7
- In `yasm` this number could be written as `10101111b`

Decimal to binary conversion

- Convert 741 to binary
- Repeatedly divide by 2 and keep the remainders

division			remainder	bits
741/2	=	370	1	1
370/2	=	185	0	01
185/2	=	92	1	101
92/2	=	46	0	0101
46/2	=	23	0	00101
23/2	=	11	1	100101
11/2	=	5	1	1100101
5/2	=	2	1	11100101
2/2	=	1	0	011100101
1/2	=	0	1	1011100101

Hexadecimal numbers

- Base 16 numbers
- Use as "digits" 0-9 and A-F (or a-f)
- A=10, B=11, C=12, D=13, E=14, F=15

$$
\begin{aligned}
\texttt{0x2b1a} &= 2 * 16^3 + 11 * 16^2 + 1 * 16 + 10 \\
&= 2 * 4096 + 11 * 256 + 16 + 10 \\
&= 8192 + 2816 + 16 + 10 \\
&= 11034
\end{aligned}
$$

Why use hexadecimal?

- Each hexadecimal digit or "nibble" is 4 bits
- `0x2b1a` = `0010 1011 0001 1010`
- `0x2b1a` = `0010101100011010b`
- Counting 32 bits for a binary pattern would be hard
- Hexadecimal is much easier
- `0xdeadbeef` = `11011110101011011011111011101111b`

Converting decimal to hexadecimal

- Convert 40007 to hexadecimal
- Repeatedly divide by 16 and keep the remainders

division			remainder	hex
40007/16	=	2500	7	7
2500/16	=	156	4	47
156/16	=	9	12	c47
9/16	=	0	9	9c47

Integers

- Integers can be 1, 2, 4 or 8 bytes long
- They can be signed or unsigned

Variety	Bits	Bytes	Minimum	Maximum
unsigned	8	1	0	255
signed	8	1	-128	127
unsigned	16	2	0	65535
signed	16	2	-32768	32767
unsigned	32	4	0	4294967295
signed	32	4	-2147483648	2147483647
unsigned	64	8	0	18446744073709551615
signed	64	8	-9223372036854775808	9223372036854775807

Negative integers

- We use the highest-order bit as a sign bit
- 1 for a sign bit means a negative number
- If we stored -1 as 10000001b
- -1 + 1 would be 10000001b + 00000001b = 100000010b
- Then addition would yield -1 + 1 = -2
- There must be a better way to store negatives
- Hopefully, we can use the same circuitry for positives and negatives

Two's complement integers

- To convert a number to its negative, use two's complement
- Flip all the bits
- Add 1
- Let's convert 1 to -1 with 8 bit numbers

```
        00000001 for the absolute value
        11111110 for the complement
        11111111 after adding 1 to the complement
   -1 = 11111111
```

- Two's complement negative numbers work for addition

More 8 bit signed integers

- They form a cycle if you keep adding 1

```
00000000 =    0
00000001 =    1
00000010 =    2
...
01111111 =  127
10000000 = -128
10000001 = -127
10000010 = -126
...
11111110 =   -2
11111111 =   -1
00000000 =    0
```

Addition

- Let's convert and add -29124 + 125

```
29124  = 0111000111000100
Negate = 1000111000111011
Add 1  = 1000111000111100

  125  = 0000000001111101

Now add  1000111000111100
         0000000001111101
         ----------------
         1000111010111001

Negate   0111000101000110
Add 1    0111000101000111
         28999

So -29124 + 125 = -28999
```

Binary multiplication

```
           1010101
*            10101
------------------
           1010101
         1010101
       1010101
------------------
       11011111001
```

Floating point numbers

- 32 bit, 64 bit and 80 bit numbers
- Stored in IEEE 754 format

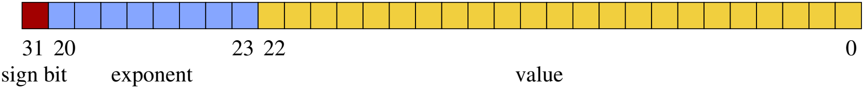
Variety	Bits	Exponent	Exponent Bias	Fraction	Precision
float	32	8	127	23	∼7 digits
double	64	11	1023	52	∼16 digits
long double	80	15	16383	64	19 digits

- Exponents are binary exponents
- An exponent field has the bias added
- A 32 bit exponent field of 128 means a binary exponent 1
- A 32 bit exponent field of 125 means a binary exponent -2
- 0.0 is stored as all bits equal to 0
- Exponent field 255 means "Not a Number"

Binary numbers with binary points

$$
\begin{aligned}
0.1_2 &= 2^{-1} \\
&= 0.5 \\
1.11_2 &= 1 + 2^{-1} + 2^{-2} \\
&= 1 + 0.5 + 0.25 \\
&= 1.75 \\
1001.1001_2 &= 2^3 + 1 + 2^{-1} + 2^{-4} \\
&= 8 + 1 + 0.5 + 0.0625 \\
&= 9.5625 \\
1.0010101 * 2^3 &= 1001.0101 \\
&= 2^3 + 1 + 2^{-2} + 2^{-4} \\
&= 8 + 1 + 0.25 + 0.0625 \\
&= 9.3125
\end{aligned}
$$

Implicit 1 bit



- Normalized floats have exponent fields from 1 to 254
- For these floats there will be at least one 1 bit in the number
- IEEE 754 uses implicit 1 bits
- For non-zero floats, they can be written in "scientific" notation
  - $1011.10101 = 1.01110101 * 2^3$
  - The leading 1 bit is not stored
  - The value (fraction) field is 01110101000000000000000
- So we have 23 bits of fraction with 1 implicit bit = 24 bits
- The sign bit is flipped to negate a float (1 means negative)

Floating point storage

- Consider consider this listing by `yasm`

```
                                      %line 1+1 fp.asm
                                      [section .data]
00000000 00000000                     zero dd 0.0
00000004 0000803F                     one dd 1.0
00000008 000080BF                     neg1 dd -1.0
0000000C 0000E03F                     a dd 1.75
00000010 0000F542                     b dd 122.5
00000014 CDCC8C3F                     d dd 1.1
00000018 F9021550                     e dd 10000000000.0
```

- The bytes are backwards
- 1.0 should be represented logically as 3F800000
- 0 sign bit, 127 exponent field, 0 for the fraction field

Floating point storage (2)

```
4 00000004 0000803F            one dd 1.0
5 00000008 000080BF            neg1 dd -1.0
6 0000000C 0000E03F            a dd 1.75
7 00000010 0000F542            b dd 122.5
```

- All these have a lot of 0 bits in the fractions
- They are all exactly equal to a sum of a few powers of 2
- $1 = 2^0$
- $1.75 = 2^0 + 2^{-1} + 2^{-2}$
- $122.5 = 2^6 + 2^5 + 2^4 + 2^3 + 2^1 + 2^{-1}$
- -1.0 differs from 1.0 only in the sign bit

Floating point storage (3)

```
8 00000014 CDCC8C3F                    d dd 1.1
```

- 1.1 is a repeating binary number
- The number in "proper" order is `3F8CCCCD`
- The exponent field is 127, so the exponent is 1
- The number is $1.00011001100110011001101_2$
- It looks like $1.1 = 1.000\overline{1100}$

Converting decimal numbers to floats

- Determine the sign bit and work with the absolute value
- Convert the whole part of the decimal number
- Convert the fraction
- Express in binary scientific notation
- Build the exponent field by adding 127 bias
- Drop the leading 1 to get the fraction field
- Example: convert -12.25
  - Sign bit is 1
  - Whole part is $12 = 1100_2$
  - Fraction is $0.25 = 0.01$
  - Scientific notation $12.25 = 1.10001_2 * 2^3$

$$
\begin{aligned}
-12.25 &= 1\ 10000010\ 10001000000000000000000 \\
&= \texttt{0xC1440000}
\end{aligned}
$$

Converting decimal number to float (2)

- The only non-obvious step is coverting the fractional part to a binary fraction.
- Suppose you have a decimal number $x = .abcdefgh$
- Then if you multiple x by 2, the only possible result is $2x < 1$ or $1 \le 2x < 2$
- If $2x < 1$, then $x < 0.5$, which means the first bit after the binary point is 0.
- If $2x \ge 1$, then $x \ge 0.5$, which means the first bit after the binary point is 1.
- So we set the first bit and work on the remaining fractional part of $2x$ to get the next bit.
- This process continues until we reach $x = 0$ or we have enough bits.

Converting decimal number to float (3)

- Let's convert -121.6875 to a binary number
- First the sign is 0
- $121 = 1111001_2$
- Now it's time to work on .6875

Multiply		Result	Binary
$.6875 * 2$	$=$	1.375	$.1_2$
$.375 * 2$	$=$	0.75	$.10_2$
$.75 * 2$	$=$	1.5	$.101_2$
$.5 * 2$	$=$	1.0	$.1011_2$

- $-121.6875 = -1111001.1011_2$
- $-121.6875 = -1.1110011011_2 * 2^6$
- As a binary float `1 10000101 11100110110000000000000`
- Expressed in hexadecimal: `0xC2F36000`

Floating point addition

- Let's add 41.275 and 0.315
- 41.275 = `101001.010001100110011010` in binary
- 0.325 = `0.0101000010100011110101110` in binary
- As with decimals, we align the numbers and add

```
   101001.010001100110011010
+       0.0101000010100011110101110
-----------------------------------
   101001.1001011100001010010101110
```

- There are 31 digits in the answer
- The answer must be rounded to 24 bits
- Rounding the last 7 bits means truncation in this case
- We get `0x42265c29` which is 41.59 (approximately)

Floating point multiplication

- Let's multiply 7.5 and 4.375

$$
\begin{array}{rrcr}
 & 7.5 & = & 111.1_2 \\
* & 4.375 & = & 100.011_2 \\
\hline
 & & & 1111_2 \\
 & & & 11110_2 \\
 & & & 111100000_2 \\
\hline
 & & & 100000.1101_2
\end{array}
$$

- Conversion to float format should be apparent by now